LINEAR CIRCUIT ANALYSIS (EED) – U.E.T. TAXILA | 09

ENGR. M. MANSOOR ASHRAF

INTRODUCTION

Thus far our analysis has been restricted for the most part to dc circuits: those circuits excited by constant or time-invariant sources.

We now begin the analysis of circuits in which the source voltage or current is time-varying.

A sinusoid is a signal that has the form of the sine or cosine function.

A sinusoidal current is usually referred to as alternating current (ac).

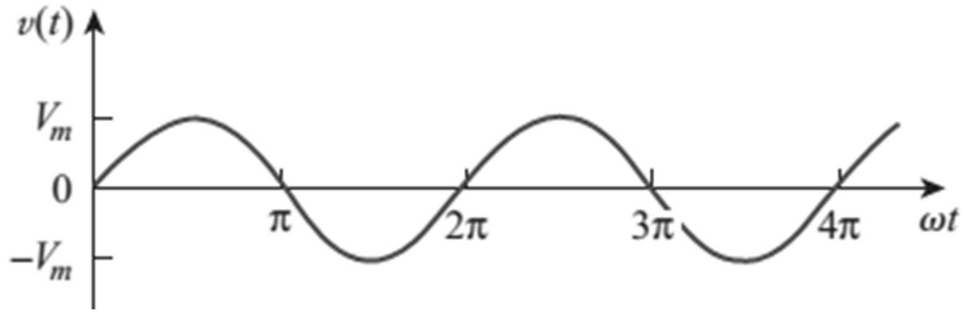
SINUSOIDS

Consider the sinusoidal voltage; $v(t) = V_m \sin \omega t$

Where V_m = the *amplitude* of the sinusoid

ω = the *angular frequency* in radians/s

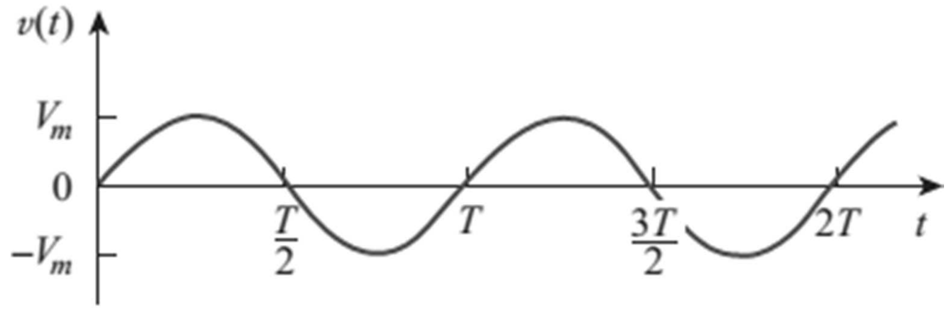
ωt = the *argument* of the sinusoid



It is evident that sinusoid repeats itself every T seconds; where T is called period of sinusoid.

SINUSOIDS

Time period of sinusoid; $T = \dfrac{2\pi}{\omega}$



A Periodic Function is one that satisfies $f(t)=f(t+nT)$, for all t and for all integers n.

For sinusoid; $v(t + T) = v(t)$

Period of periodic function is time for complete cycle.

SINUSOIDS

The reciprocal of time period is number of cycles per second, known as the Cyclic Frequency, measured in hertz (Hz).

$$f = \frac{1}{T} \qquad \omega = 2\pi f$$

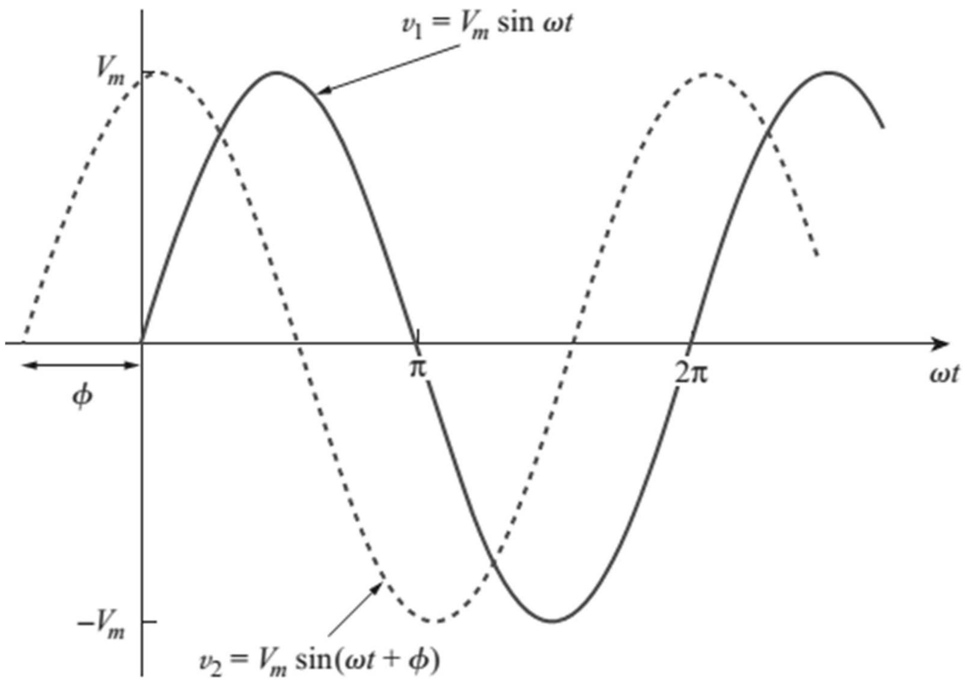
Consider general sinusoid;

$$v(t) = V_m \sin(\omega t + \phi)$$

Lets examine two sinusoid;

$$v_1(t) = V_m \sin \omega t \qquad v_2(t) = V_m \sin(\omega t + \phi)$$

SINUSOIDS



SINUSOIDS

The sinusoid can be expressed in either sine or cosine form.

Following trigonometric identities may be used;

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

To interchange sine and cosine functions;

$$\sin(\omega t \pm 180^\circ) = -\sin\omega t$$
$$\cos(\omega t \pm 180^\circ) = -\cos\omega t$$
$$\sin(\omega t \pm 90^\circ) = \pm\cos\omega t$$
$$\cos(\omega t \pm 90^\circ) = \mp\sin\omega t$$

PROBLEMS

Find amplitude, phase, period and frequency of sinusoid: $v(t) = 12\cos(50t + 10^\circ)$

(12 V, 10°, 50 rad/s, 0.1257 s, 7.958 Hz)

Calculate the phase angle between two sinusoids and state which sinusoid is leading?

$$v_1 = -10\cos(\omega t + 50^\circ)$$
$$v_2 = 12\sin(\omega t - 10^\circ)$$

(30°,v_2)

PHASORS

A Phasor is a complex number that represents the amplitude and phase of a sinusoid.

A complex number can be represented in three forms as;

$$z = x + jy \qquad \text{Rectangular form}$$

$$z = r\angle\phi \qquad \text{Polar form}$$

$$z = re^{j\phi} \qquad \text{Exponential form}$$

The relationship between the rectangular form and the polar form is;

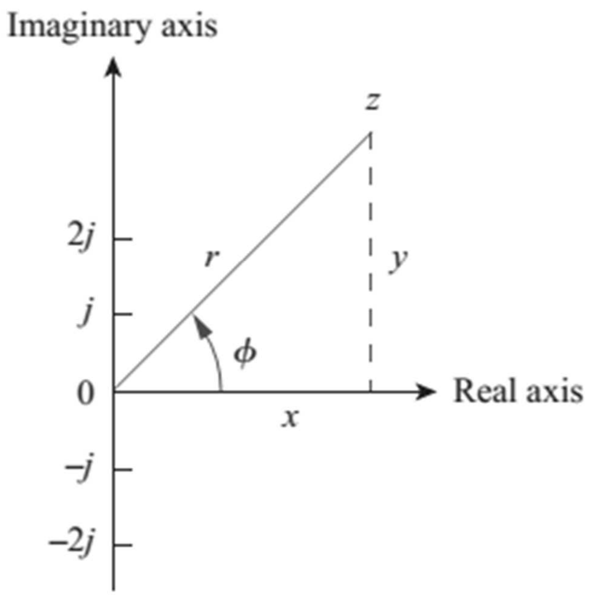
$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$

PHASORS

The complex may be written and interpreted as;

$$z = x + jy = r\angle\phi = r(\cos\phi + j\sin\phi)$$



PHASORS

Addition and subtraction of complex numbers are better performed in rectangular form while multiplication and division in polar form.

$$z = x + jy = r\angle\phi, \qquad z_1 = x_1 + jy_1 = r_1\angle\phi_1$$

$$z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

Addition; $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction; $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication; $z_1 z_2 = r_1 r_2 \angle\phi_1 + \phi_2$

Division; $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\angle\phi_1 - \phi_2$

PHASORS

Reciprocal;

$$\frac{1}{z} = \frac{1}{r}\angle-\phi$$

Square Root; $\sqrt{z} = \sqrt{r}\,\angle\phi/2$

Complex Conjugate;

$$z^* = x - jy = r\angle-\phi = re^{-j\phi}$$

In complex numbers;

$$\frac{1}{j} = -j$$

PHASORS

The idea of phasor representation is based on Euler's identity.

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

where;

$$\cos\phi = \mathrm{Re}(e^{j\phi})$$

$$\sin\phi = \mathrm{Im}(e^{j\phi})$$

Given a sinusoid; $v(t) = V_m \cos(\omega t + \phi)$

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t + \phi)})$$

$$v(t) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t})$$

$$v(t) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

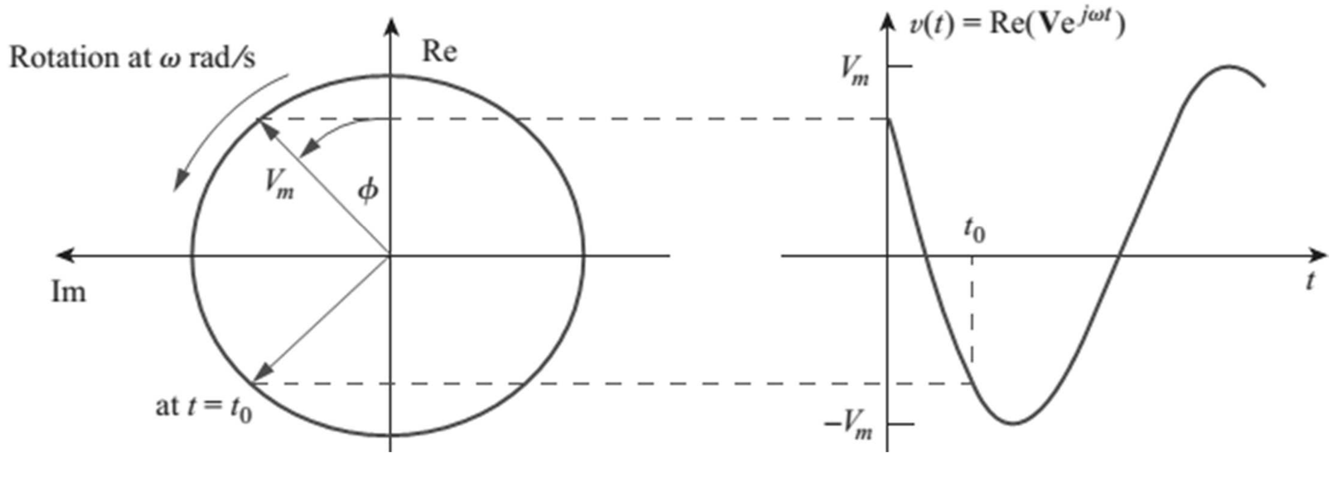
where;

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle\phi$$

PHASORS

'**V**' is the phasor representation of the sinusoid v(t).

The phasor rotates in complex plane to plot sine wave in time domain using its projection.



PHASORS

The transformation is summarized as;

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \angle \phi$$

(Time-domain representation) (Phasor-domain representation)

Time domain and phasor domain representation can be interchanged as;

Time domain representation	Phasor domain representation
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90°$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90°$

PHASORS

Time domain is also known as Instantaneous domain and phasor domain is also known as Frequency domain.

Consider sinusoid;

$$v(t) = \text{Re}(\mathbf{V}e^{j\omega t}) = V_m \cos(\omega t + \phi)$$

Differentiating;

$$\frac{dv}{dt} = -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi + 90°)$$

$$= \text{Re}(\omega V_m e^{j\omega t} e^{j\phi} e^{j90°}) = \text{Re}(j\omega \mathbf{V} e^{j\omega t})$$

Transformation of derivative from time domain to phasor domain is;

$$\frac{dv}{dt} \quad \Leftrightarrow \quad j\omega \mathbf{V}$$

(Time domain) (Phasor domain)

PHASORS

Transformation of integral from time domain to phasor domain is;

$$\int v\, dt \quad \Leftrightarrow \quad \frac{\mathbf{V}}{j\omega}$$

(Time domain) (Phasor domain)

The v(t) is time dependent, while **V** is not.

The v(t) is always real with no complex term, while **V** is generally complex.

PROBLEMS

Evaluate these complex numbers:

(a) $(40\angle 50^\circ + 20\angle -30^\circ)^{1/2}$

(b) $\dfrac{10\angle -30^\circ + (3 - j4)}{(2 + j4)(3 - j5)^*}$

Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40^\circ)$ A
(b) $v = -4\sin(30t + 50^\circ)$ V

Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A
(b) $\mathbf{V} = j8e^{-j20^\circ}$ V

PROBLEMS

Given $i_1(t) = 4\cos(\omega t + 30°)$ A and $i_2(t) = 5\sin(\omega t - 20°)$ A, find their sum.

$i(t) = 3.218\cos(\omega t - 56.97°)$ A

Using the phasor approach, determine the current $i(t)$ in a circuit described by the integrodifferential equation

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75°)$$

$i(t) = 4.642\cos(2t + 143.2°)$ A

PHASOR RELATIONSHIPS FOR CIRCUIT ELEMENTS

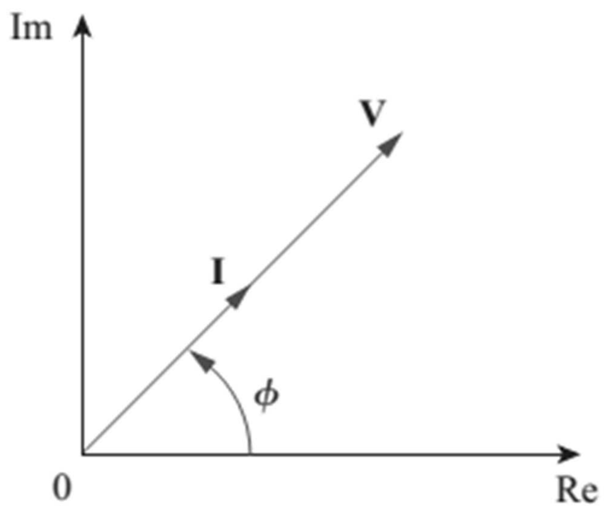
If the current through the resistor R is $i=I_m cos(\omega t+\phi)$, voltage across resistor is given by Ohm's law.

$$v = iR = RI_m\cos(\omega t + \phi)$$

Phasor form;

$$\mathbf{V} = RI_m\angle\phi$$

$$\mathbf{V} = R\mathbf{I}$$



PHASOR RELATIONSHIPS FOR CIRCUIT ELEMENTS

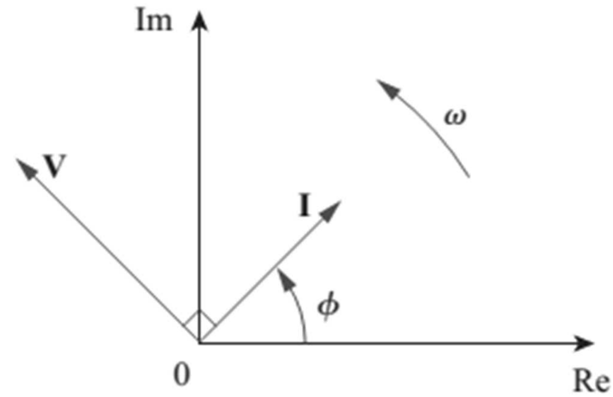
If the current through the inductor L is $i=I_m cos(\omega t+\phi)$, voltage across inductor is given by;

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi) \quad v = \omega L I_m \cos(\omega t + \phi + 90°)$$

$$\mathbf{V} = \omega L I_m e^{j(\phi+90°)} = \omega L I_m e^{j\phi} e^{j90°} = \omega L I_m \angle \phi + 90°$$

$$e^{j90°} = j$$

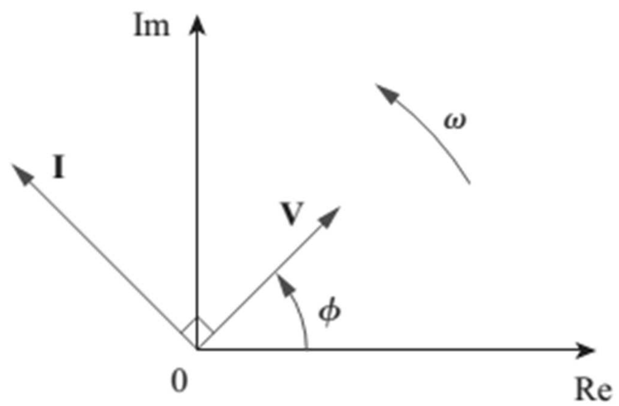
$$\mathbf{V} = j\omega L \mathbf{I}$$



PHASOR RELATIONSHIPS FOR CIRCUIT ELEMENTS

If the voltage across capacitor C is $v=V_m cos(\omega t+\phi)$, current through capacitor is given by;

$$i = C\frac{dv}{dt} \qquad \mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$



PHASOR RELATIONSHIPS FOR CIRCUIT ELEMENTS

Summary

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

PROBLEMS

The voltage $v = 12 \cos(60t + 45°)$ is applied to a 0.1-H inductor. Find the steady-state current through the inductor.

$i(t) = 2 \cos(60t - 45°)$ A

If voltage $v = 10 \cos(100t + 30°)$ is applied to a 50 μF capacitor, calculate the current through the capacitor.

$50 \cos(100t + 120°)$ mA

IMPEDANCE AND ADMITTANCE

The voltage across resistor, inductor and capacitor are;

$$\mathbf{V} = R\mathbf{I}, \qquad \mathbf{V} = j\omega L\mathbf{I}, \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \qquad \text{or} \qquad \mathbf{V} = \mathbf{Z}\mathbf{I}$$

Where Z is a frequency dependent quantity known as impedance, measured in ohms.

The Impedance of a circuit is the ratio of the phasor voltage to the phasor current, measured in Ohms.

IMPEDANCE AND ADMITTANCE

The Admittance (Y) is the reciprocal of impedance, measured in Siemens.

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}$$

Impedances and admittances of passive elements.

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \frac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \frac{1}{j\omega L}$
C	$\mathbf{Z} = \frac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

IMPEDANCE AND ADMITTANCE

The impedance may be expressed in rectangular form;

$$\mathbf{Z} = R + jX$$

$R = \text{Re}\,\mathbf{Z}$ is the *resistance* $X = \text{Im}\,\mathbf{Z}$ is the *reactance*

Reactance may be positive or negative depending on circuit element.

Inductive reactance for inductor;

$$\mathbf{Z} = R + jX$$

Capacitive reactance for capacitor;

$$\mathbf{Z} = R - jX$$

IMPEDANCE AND ADMITTANCE

The impedance may be expressed in polar form;

$$\mathbf{Z} = |\mathbf{Z}| \angle\theta$$

$$\mathbf{Z} = R + jX = |\mathbf{Z}| \angle\theta$$

Relationship between rectangular and polar forms of impedance.

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |\mathbf{Z}|\cos\theta, \qquad X = |\mathbf{Z}|\sin\theta$$

IMPEDANCE AND ADMITTANCE

Admittance is reciprocal of impedance and is combination of Conductance and Susceptance as real and imaginary parts.

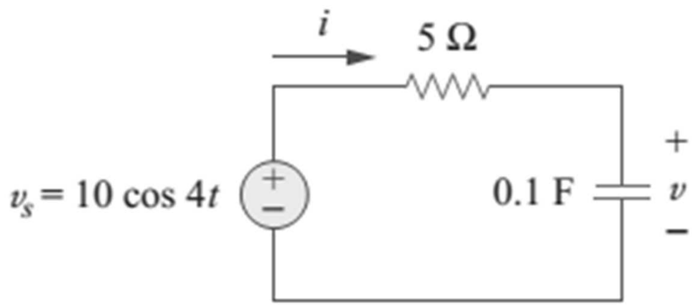
$$\mathbf{Y} = G + jB$$

$$G + jB = \frac{1}{R + jX}$$

$$G = \frac{R}{R^2 + X^2}, \qquad B = -\frac{X}{R^2 + X^2}$$

PROBLEMS

Find v(t) and i(t)?



KIRCHHOFF'S LAWS IN FREQUENCY DOMAIN

KVL holds in phasor domain as in time domain.

$$v_1 + v_2 + \cdots + v_n = 0$$

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

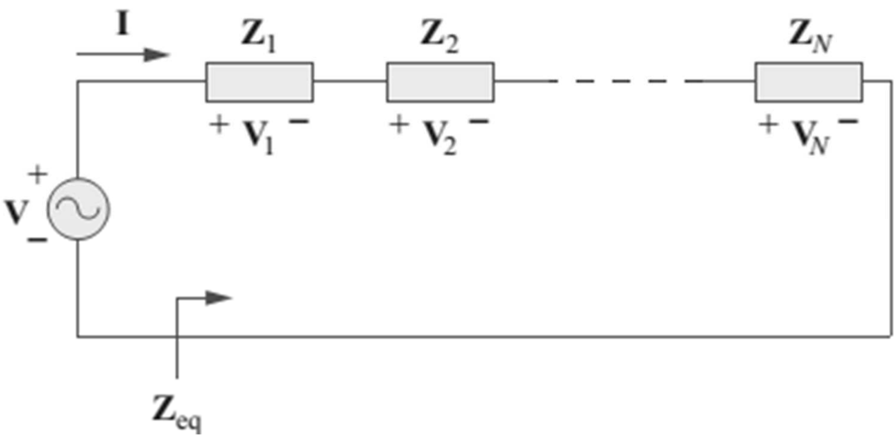
KCL also holds in phasor domain as in time domain.

$$i_1 + i_2 + \cdots + i_n = 0$$

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0$$

IMPEDANCE COMBINATIONS

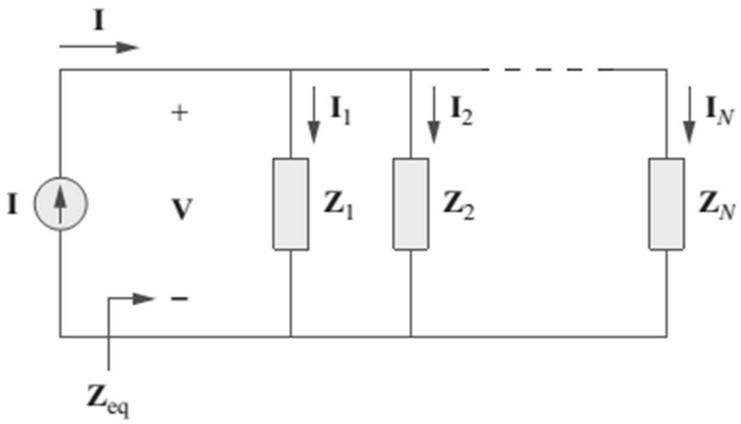
Equivalent impedance in series circuit;



$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

IMPEDANCE COMBINATIONS

Equivalent impedance in parallel circuit;



$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

IMPEDANCE COMBINATIONS

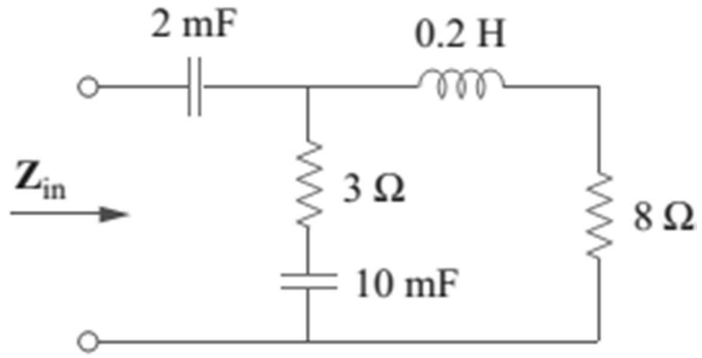
Voltage and current division rules are also equally applicable for series and parallel circuits;

$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

PROBLEMS

Find equivalent impedance at ω=50 rad/s.



REFERENCES

Fundamentals of Electric Circuits (4th Edition)

Charles K. Alexander, Matthew N. O. Sadiku

Chapter 09 – Sinusoids and Phasors (9.1 – 9.7)

Exercise Problems: 9.1 – 9.70

Do exercise problem yourself.